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LOCATION-ROUTING PROBLEMS

by

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ABSTRACT

Location-routing problems involve simultaneously locating a number of facilities among candidate sites and establishing delivery routes to a set of users in such a way that the total system cost is minimized. This paper presents a survey of such problems. The main examples and applications found in the literature are described. Heuristic and exact algorithms for their solution are then presented.

KEY WORDS : location, vehicle routing, location-routing problems, travelling salesman problem

RÉSUMÉ

Les problèmes simultanés de localisation et de construction de tournées consistent à déterminer, dans un ensemble de sites potentiels, un certain nombre d'emplacements à partir desquels on établira des tournées de véhicules devant visiter des usagers. L'objectif consiste à minimiser le coût total du système. On présente dans cet article une revue de ces problèmes. L'article fait état des principaux exemples et applications décrits dans la littérature scientifique. On passe ensuite en revue les principaux algorithmes approximatifs et exacts pour le problème étudié.

MOTS CLÉS : localisation, problèmes de tournées, problèmes mixtes de localisation et de construction de tournées, problème du voyageur de commerce

TABLE OF CONTENTS

1. Introduction. 1

1.1 Two main components of distribution management 1
1.2 On the need to construct specific location-routing algorithms 2
1.3 A general framework. 5
1.4 Organization of the paper. 8

2. Applications and examples. 9

2.1 Two-layer applications : M(T). 9
2.2 Three-layer applications : M(R,T). 11
2.3 Three-layer applications : M(T,R). 12
2.4 A three-layer application : M(T,T) 14
2.5 Four-layer applications : M(R,R,T) 14

3. Heuristic algorithms. 15

3.1 Typology 15
3.2 LAR algorithms 17
3.3 ARL algorithms 20

4. Exact algorithms for the vehicle routing problem. 21

4.1 Classification 21
4.2 Direct search algorithms 22
4.3 Dynamic programming. 22
4.4 Set partitioning formulations. 23
4.5 Flow formulations. 24

5. Three-index vehicle flow formulations 25

5.1 Objective, constraints and notation. 26
5.2 Formulation. 28
5.3 Special cases. 30

6. Two-index vehicle flow formulations and algorithms for symmetrical problems. 30

6.1 A classical TSP formulation and some extensions. 31
6.2 The single depot LRP 33
6.3 A family of multi-depot LRPs 34
6.4 A multi-depot capacitated LRP. 37

7. Two-index vehicle flow formulations and algorithms for asymmetrical problems. 41

7.1 Problem definition 41
7.2 Graph extension. 42
7.3 Interpretation 44
7.4 The model. 45
7.5 Algorithm. 48

8. Conclusion. 48

REFERENCES. 50

LIST OF FIGURES

Figure 1 : A three-layer distribution system. 6
Figure 2 : Feasible solution on G'. 46
Figure 3 : Feasible solution on G 47

1. INTRODUCTION

1.1 Two main components of distribution management

Distribution management problems traditionally include two main components :

i) **location** : determine the optimal location of facilities in order to minimize the weighted sum of distances between these facilities and a set of users (Rapp, 1962; Cooper, 1963, 1964; Hakimi, 1964, 1965) or the maximum distance between the facilities and the users (Minieka, 1970) or a combination of these two objectives (Halpern, 1976);

ii) **routing** : determine optimal collection or delivery routes from a set of depots to a set of users, subject to some restrictions, the most common being :

- every user is served by only one depot but several users may be on the same route;
- vehicles start and end their routes at the same depot;
- routes can be constrained by maximum length or weight considerations, time windows for specific visits, precedence relations between users, etc.

A description of these restrictions can be found in Christofides (1985) and in Bodin and Golden (1981).

Considerable progress has been made in each of these two areas over the last twenty-five years. In location theory, this evolution can be traced back to the seminal papers of Rapp (1962) and Cooper (1963, 1964) on location-allocation problems, and to Hakimi's (1964, 1965) work on median problems. Over the last ten years, the field has grown considerably, as witnessed by the holding of four international symposia on location theory (Banff, 1978; Skodsborg, 1981; Boston, 1984 and Namur, 1987), which have led to the publication of three special issues of the European

Journal of Operational Research (vol. 6, no. 1 (1981); vol. 12, no. 1 and 3 (1983)) and to a book edited by Osleeb and Ratick (1986). Other recent comprehensive works include the books by Handler and Mirchandani (1979) and by Francis and Mirchandani (1987), Labbé's thesis (1985), the bibliography by Domschke and Drexl (1985) and the surveys of Francis et al. (1983), Tansel et al. (1983a, 1983b), Aikens (1985) and Hansen et al. (1987).

Most research on vehicle routing stems from the study of the travelling salesman problem (TSP) which consists of determining the shortest Hamiltonian circuit through n nodes, users or cities. The first major publication on the TSP appeared more than three decades ago (Dantzig et al., 1954). Work on the TSP and on several variants of the vehicle routing problem (VRP) has since flourished with the advent of the first branch and bound algorithms (Land and Doig, 1960; Little et al., 1963) and with the development of cutting planes for integer linear programming (ILP) (Gomory, 1963). Recent progress can be attributed to the refinement and use of Lagrangean relaxation techniques (Fisher and Jaikumar, 1978, 1981; Christofides et al., 1981a; Balas and Christofides, 1981; Desrosiers et al. 1986b), to the development of state-space relaxation in dynamic programming (Christofides et al., 1981b) and to the use of constraint relaxation techniques (Laporte et al., 1985). Comprehensive surveys on vehicle routing include those of Christofides et al. (1979), Mole (1979), Watson-Gandy and Foulds (1981), Magnanti (1981), Bodin et al. (1983), Christofides (1985), Laporte and Nobert (1987).

1.2 On the need to construct specific location-routing algorithms

There exist several practical situations where the need arises to locate one or several facilities and to construct associated (delivery) routes including several users. In such contexts, location and routing are

intertwined decisions which must be modelled and optimized simultaneously. These **location-routing problems** (LRPs) belong to the general class of network optimization problems (Golden et al., 1981) and to the family of arc-node problems (Schrage, 1981) since their solution can be described by a selection of arcs and nodes in a network. Research on LRPs is relatively new and the literature on it is much less developed than that associated with either pure location or pure routing problems. This can be explained partly by the fact that the need to combine the two subjects has been recognized only fairly recently (Webb, 1968; Christofides and Eilon, 1969) and partly because location-routing problems are in general very difficult to solve (Rand, 1976, p. 248; Handler and Mirchandani, 1979, p. 189). It is therefore not surprising that most algorithms for LRPs are approximate; only recently have exact procedures been proposed and most of these can only solve very special classes of LRPs.

One possible approach to tackling location-routing problems is to treat them as pure location problems in which the (marginal) cost of making a delivery to a user from a given facility is approximated by a linear function of the radial distance between the user and the facility. Webb (1968) pointed out that such an approach is inadequate : "A small increase to an existing delivery or collection quantity may add an insignificant and undetectable amount to the actual transport cost if the vehicle has spare capacity, but might add the cost of a special journey if there is no spare capacity. Similarly, an extra delivery or collection may cost virtually nothing if a vehicle with adequate spare capacity is already due to pass, or as much as the cost of a special journey if not. Knowledge of the size and position of a delivery or collection is insufficient to specify the basic transport cost incurred" (p. 312). After some numerical experiments, Webb concludes : "The relationship between planned distance [author's note : length of optimal tour through all users served by the same vehicle] (...) and either moment or distance sums (...) is not

always very good". Moreover : "The point of minimum planned distance did not coincide with the point of minimum moment sum." (p. 315). (This observation was later confirmed by Laporte and Nobert (1981) in a series of tests.) The same question was also investigated by Christofides and Eilon (1969) who examined the case of n customers randomly and uniformly distributed in a square of side a. These are served by m vehicles based at the same facility located within the square and it is further assumed that the m routes do not intersect. Letting z* denote the optimal total length of the m routes and z_R, the sum of the radial distances between the customers and the facility, they showed that z* can be approximated by the following formula :

$$z^* \triangleq Amz_R/n + B\sqrt{a}\sqrt{z_R} \qquad (1)$$

where A and B are two constants related to the position of the facility within the square. This result was derived from the asymptotic formula for the expected optimal length of a travelling salesman route in the plane, developed by Bearwood et al. (1959).

When the first term of (1) predominates over the second term, i.e. when

$$z_R \gg a\left(\frac{Bn}{Am}\right)^2 \qquad (2)$$

it is valid to approximate z* by a multiple of z_R. Further developments based on this analysis can be found in Eilon et al. (1971).

When condition (2) is not satisfied, the need to develop specific approaches to LRPs remains. It may also exist even when (2) holds since

i) customers are not always uniformly distributed in the plane;
ii) the value of m may not be known a priori;
iii) routes may intersect;
iv) various side constraints which were not taken into account in the development of (1) sometimes exist;
v) the facility is sometimes located outside the customer area.

1.3 A general framework

If we consider users or customers on one hand, and facilities to be located on the other hand, we can theoretically arrive at the definition of four different types of problems depending on whether users or facilities are assumed to belong to discrete or continuous sets. Like most authors in the fields of location and routing, we will assume that the user set is always discrete (problems with continuous user sets have been described to a large extent in the literature on Voronoi diagrams - see Eiselt and Pederzoli (1986a, 1986b) for a recent survey). Moreover, as in most studies, the facility set is also discrete, we will restrict ourselves to this case in the presentation of the general framework. We also temporarily assume that users and facilities belong to disjoint sets but some examples where this assumption does not hold will be presented.

Several distribution systems can be represented by a λ-layer diagram such as the one depicted by figure 1. In this example, there are three layers which have been identified as **primary facilities, secondary facilities** and **users.** Frequently, the primary facilities will represent factories, secondary facilities will correspond to depots or warehouses and users will be customers. Primary facilities are usually situated at known and fixed locations and so are users. On the other hand, the location of secondary facilities will frequently not be determined a priori : the number or location of these facilities, together with the associated distribution routes constitute decision variables. There can of course be only two layers (as in the transportation problem (Hitchcock, 1941), the classical location-allocation problem (Cooper, 1963), the p-median problem (Hakimi, 1964), the vehicle routing problem (Laporte and Nobert, 1987)) or more than three layers (Mercer et al., 1978; Dejax, 1986; Dejax and Turri, 1986).

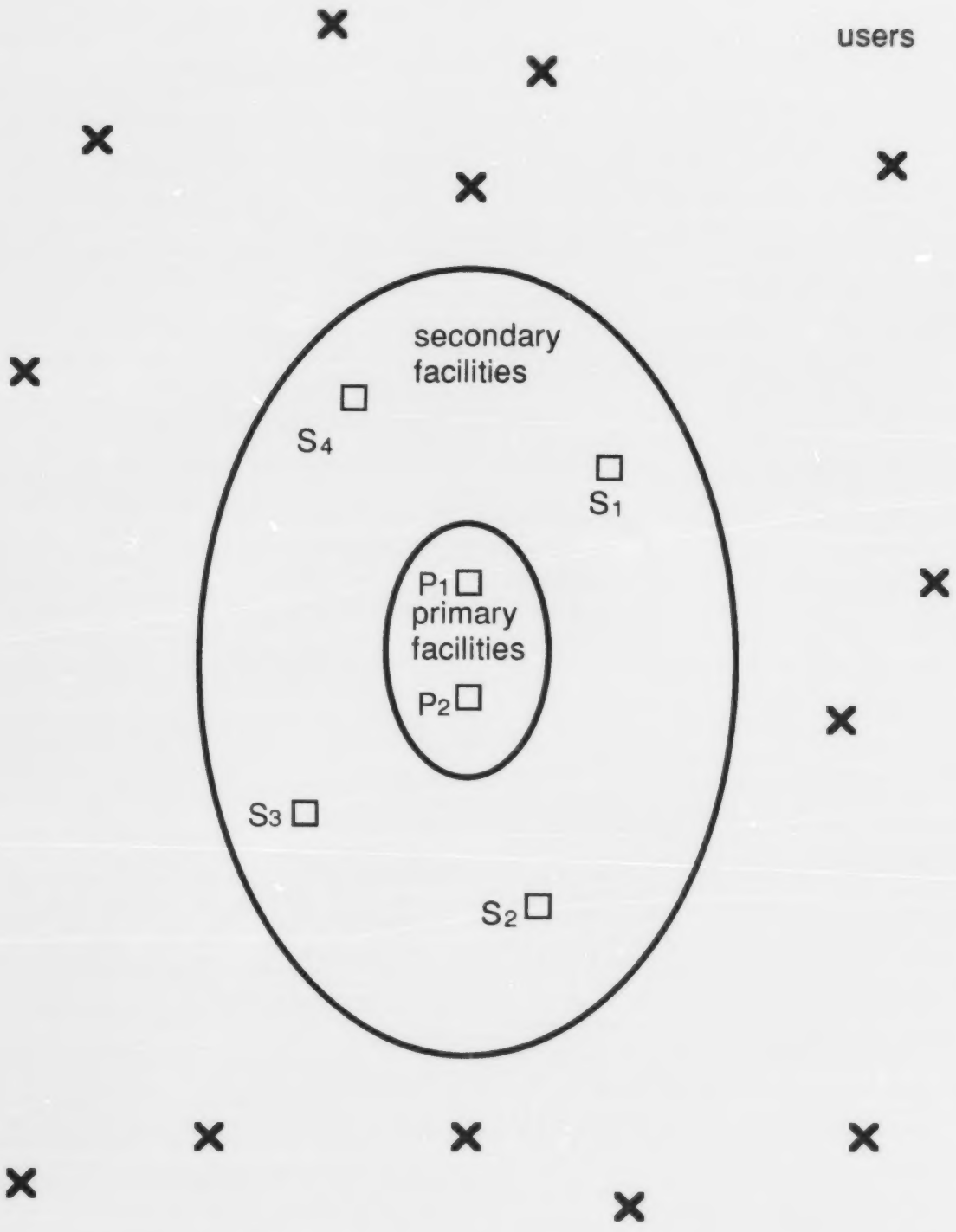

Figure 1

A three-layer distribution system

Several problems can be defined according to the distribution mode M_t used by vehicles based at a facility located at layer t. Usually, these vehicles will make trips to layer t + 1, although some authors (e.g. Watson-Gandy and Dohrn (1973)) consider the possibility of making direct trips from layer 1 to layer 3. We consider two distribution modes :

M_t = R : all trips from layer t must be return trips (i.e. trips to and from a single user or facility);

M_t = T : trips from layer t may be tours (i.e. round-trips through several users or facilities) or return trips.

The distribution modes used for the whole system will be represented by the expression $M(M_1,\ldots,M_{\lambda-1})$. Thus, in a three-layer system, we will have the four following possibilities :

M(R,R) : occurs mostly when shipments of a generally bulky material (lumber, cement) have to be made in full loads between successive layers.

M(R,T) : here, large shipments arriving at the secondary facilities are broken up and dispatched in smaller loads to customers. This situation is encountered in the food industry, for example.

M(T,R) : in this situation, trips are often made from the users who bring goods to the secondary facilities (Labbé and Laporte, 1986; Nambiar and Chalmet, 1979; Nambiar et al., 1981): these goods are then collected in round-trips and brought to primary facilities.

M(T,T) : this case is frequently encountered in the newspaper industry : here, the primary facilities are printing plants; newspapers are dispatched daily to secondary facilities (transfer points) and then again to retail outlets (see Jacobsen and Madsen, 1978, 1980).

In location-routing problems,

i) locational decisions must be made for at least one layer (otherwise, the problem reduces to a pure routing problem);

ii) tours (M_t = T) must be allowed at least once (otherwise the problem becomes a pure location problem).

The most common objective in LRPs is to minimize a linear combination of (i) routing costs, (ii) vehicle fixed costs and (iii) depot operating costs. These have to be scaled down so that they relate to the same time horizon. Some authors however (see for example Watson-Gandy and Dohrn, 1973; Mercer, 1970, 1978; Mercer et al., 1978) note the danger of locating distribution centers without considering the effect on sales revenues of their proximity to customers. Customers located far from distribution centers tend to be less profitable. Finally, problems where costs are shared between users and the organization providing service yield multicriteria objectives (see the case M(T,R) above). Several approaches are available for such problems; one possible strategy is to combine the various criteria into a single objective (Labbé and Laporte, 1986).

If we now turn to the various constraints involved in location-routing, a wide variety of problems can arise. (We are dealing here with a class of problems rather than with a specific problem.) For the sake of clarity and of simplicity, these and the associated notation will be presented in the course of the text, as the need arises.

1.4 Organization of the paper

The remainder of this paper is organized as follows. In the next section, we present some applications and examples of LRPs. Section 3 contains a description of the main heuristics developed for LRPs. In section 4, we review briefly the various algorithms used for the exact solution of VRPs since these have inspired most research on the development of exact algorithms for LRPs. Then follows, in sections 5, 6 and 7 a classification and review of the various ILP formulations and algorithms for LRPs :

- section 5 : three-index vehicle flow formulations;
- section 6 : two-index vehicle flow formulations and algorithms for symmetrical problems;
- section 7 : two-index vehicle flow formulations and algorithms for asymmetrical problems.

The conclusion is presented in section 8.

2. APPLICATONS AND EXAMPLES

The general framework description of section 1.3 was left sufficiently general so that it would account for most problems involving at some stage simultaneous location and routing decisions. Such problems arise in a variety of practical situations. We briefly review in this section some of the main applications described in the operational research literature. Two classes of problems, the generalized travelling salesman problem and the hierarchical network design problem are also presented as LRPs. The section is divided according to the $M(M_1,\ldots,M_{\lambda-1})$ classification introduced in section 1.3.

2.1 Two-layer applications : M(T)

Two-layer problems constitute the simplest class of LRPs. These usually involve the optimal location of one or several depots and the design of optimal delivery routes to customers. One such example, described by Or (1976) and later by Or and Pierskalla (1979), concerns the optimal location of blood banks. The problem consists of determining the locations of a fixed number central blood banks among candidate sites, the optimal allocation of hospitals to these banks and the optimal blood delivery routes to hospitals. These routes are constrained by vehicle capacities and maximum number of daily deliveries. The authors studied fifteen cases in the Chicago area. In one example, representative of these cases, there were 3 blood banks, serving a total of 117 hospitals, with

one vehicle based at each bank. In a second example derived from the first, hospitals were used as blood bank locations, leaving 114 hospitals to be served by a total of 8 vehicles.

Burness (1974), Burness and White (1976) and Drezner et al. (1984a, 1984b) describe the problem of selecting a central depot location among n points so that the expected length of a tour through the depot and a subset of randomly selected points is minimized. Applications mentioned by Burness and White include the location of a storeroom in an industrial plant which delivers orders to departments located throughout the plant area, the location of a district sales office when the salesman services a number of customers and the location of a truck depot which provides pick-up and delivery service for a number of firms.

Chan and Francis (1976) analyze the problem of optimally locating a depot in a tree in order to minimize the maximum distance travelled by n vehicles making two deliveries each. Chan and Hearn (1977) consider the same problem, but in the plane. Ghosh et al. (1981) show that the optimal location is sensitive to the choice of distance norm. Ichimori and Nishida (1985) and Drezner (1985) provide exact O(n log n) algorithms for the Chan and Hearn problem. Finally, Kolen (1985) studies the problem of locating several depots in a tree, under the same assumptions, in order to minimize the maximum round-trip cost. He then studies the problem of determining the minimum number of depots required in order to ensure that the length of a round-trip does not exceed a given bound. Both problems are solved in polynomial time.

Laporte and Louveaux (1986) studied the problem of locating a central depot and of establishing several delivery routes for vehicles of given capacity to customers with a variable demand. Two variants of the problem are considered :

i) the probability that the demand of a planned route exceeds the vehicle capacity is bounded by a preset value;

ii) penalties associated with "route failures" are computed; the planned routes must be such that their expected penalty does not exceed a preset treshold. This problem constitutes an extension of pure stochastic-routing problems treated by Stewart and Golden (1983) and Dror and Trudeau (1986), for example.

McLain et al. (1984) describe a military application which consists of determining, among a set of NATO bases, a number of operating locations for aircraft, in such a way that the total flying time of all circular routes through these locations and a number of military bases is minimized. The case studied by the authors involves 5 operating locations, 84 military bases, and between 14 and 18 routes, depending on the solution which is adopted.

The generalized travelling salesman problem (GTSP) was originally presented by Henry-Labordere (1969) and Srivastava et al. (1969) and recently reexamined by Laporte et al. (1983b, 1987). It can be stated as follows : given a depot and several sets of nodes (which may intersect), determine the shortest Hamiltonian circuit through the depot and at least one node from every set. This problem may be viewed as an M(T) location-routing problem with a single node at the first layer and sets of potential locations at the second layer : at least one location must be chosen within each set. This problem has several applications, e.g. the optimal location of post boxes in an urban area (Bovet, 1983). A constrained version of the GTSP was recently studied by Laporte et al. (1986b).

2.2 Three-layer applications : M(R,T)

This class of problems is generally associated with the classical three-layer set-up (factories, depots, customers), with trunk deliveries

from factories to depots and round-trips to serve the customers from the depots. Location decisions are usually made at the second layer.

The study described by Watson-Gandy and Dohrn (1973) arises from the food and drink industry. It was initiated at the request of a company operating from 5 factories and 11 depots in Southern England and Wales. Initially, the problem consisted of relocating two of the depots by considering the costs involved and the impact on sales levels. The authors recommended closing 3 of the existing depots and of opening 8 new ones, resulting in a net profit increase of £ 60,000 per year.

Von Bednar and Strohmeier (1979) describe a large scale depot location study undertaken for an Austrian manufacturer of consumer goods. The problem involved several customers (aggregated into 1,500 groups). The results were implemented.

Finally, the TFD paper (1977) examines the location of depots among 13 candidate sites, in order to deliver goods to 100 shops in Sweden. Few details are provided on the solution. [The last two examples were summarized from Madsen (1983).]

2.3 Three-layer applications : M(T,R)

This type of model differs from most others in that travelling costs between the first and second layers and between the second and third layers are usually shared by two different entities, e.g. a vehicle fleet operator and users.

Nambiar et al. (1981) describe such an example. It is based on a previous study by Nambiar and Chalmet (1979). In Malaysia, smallholers account for approximately 53% of total natural rubber production. In order to improve the quality of smallholder rubber and to enhance its marketing structure, the Malaysian government recently began the establishment of central rubber processing factories (CRPFs). Rubber collection is done in

two stages : (i) smallholders first bring their crop to collection stations (CS). Each CS serves between 70 and 80 smallholders within a radius of 1 to 3 km; (ii) vehicles based at the CRPFs then tour a total of about 50 CSs where they collect the rubber. All tours are kept within 10 hours in order to ensure that the latex does not precoagulate before arriving at the CRPF. The problem considered by Nambiar et al. consists of determining the optimal CRPF locations (all CSs are potential sites) as well as the associated collection routes.

The problem treated by Labbé and Laporte (1986) arises from a different context, but is similar in many ways to the rubber collection problem. It concerns the optimal location of post boxes in an urban or a rural area. The authors consider an area divided into zones, each containing a number of known and potential post box sites. Within a planning horizon, all users travel a given expected distance to the nearest post box located within their zone. Postal vehicles are dispatched regularly from a postal post office to collect the post boxes. The problem consists of determining the post box locations which will minimize a linear combination of user inconvenience (total expected travel distance) and post office costs. The main structural difference between this problem and the previous one is that here, location decisions have to be made about facilities of the second layer, as opposed to the first layer in the natural rubber example.

Hierarchical network design problems (Current et al., 1985, 1986) may also be viewed as M(T,R) LRPs. The problems consist of locating, in a network, a primary path with predetermined starting and terminus nodes or a closed path, as well as secondary paths linking users to the primary path. Such problems occur in the design of expressway and feeder road networks, for example. Similar applications arise in the optimal location of bus routes (Vandale, 1985) : as in the post box problem, consideration

must be given to the cost of operating the bus route and to user inconvenience (walking to and from the closest bus stop).

2.4 A three-layer application : M(T,T)

Jacobsen and Madsen (1978, 1980) describe an M(T,T) location-routing problem related to the daily distribution of newspapers in Denmark. Each day, approximately 150,000 newspapers are dispatched by primary vehicles from the printing office to transfer points where they are loaded into secondary vehicles for final delivery to some 4,500 retailers. Both primary and secondary vehicles make round-trips which must be of limited duration if newspapers are to reach the retailers no later than a certain time. The problem considered by Jacobsen and Madsen consists of determining optimal locations for the transfer points as well as primary and secondary tours.

2.5 Four-layer applications : M(R,R,T)

In M(R,R,T) problems, the four layers are usually factories, warehouses, depots and customers. Goods are dispatched in full loads up to the third layer; routes to customers are usually round-trips. Decisions have to be made on the number and location of warehouses and depots, as well as on delivery routes. Two examples of this arrangement are provided by Mercer et al. (1978, chapters 2 and 3).

The first case study describes the operations of a consumer durables company in England. The authors weighed the advantages of having a central warehouse and several depots serving as transshipment points, as opposed to a larger number of warehouses (13 before the study was undertaken) serving the customers directly. In making their recommendations, the authors took into consideration such factors as (i) capital and operating costs; (ii) lower inventory costs, discount from manufacturers and better control of service and cost associated with a centralized system;

(iii) probabilities of stockouts and late deliveries to retailers (customers) and lost sales resulting from stockouts. The final recommendation was to open one central warehouse and 23 depots.

The second study was undertaken for Cadbury-Schweppes in the United Kingdom. In a sense, it is similar to the previous one : the authors wished to determine the best location of depots (second layer) and sub-depots (third layer) to serve approximately 50,000 customers grouped together in about 500 zones. Two solutions were to be provided : determine the best configuration (i) starting from scratch; (ii) given the present pattern of depots. The proposed distribution system consisted of 7 depots located in major U.K. cities and of 23 sub-depots, resulting in yearly savings of £ 125,000. The authors also recommended ways of moving from the company's existing situation to the green-field system. A follow-up investigation indicated that about one third of potential savings had in fact been achieved.

3. HEURISTIC ALGORITHMS

3.1 Typology

LRPs are often described as a combination of three distinct components : (i) facility **location**; (ii) **allocation** of users to facilities; (iii) vehicle **routing** (see for example McLain et al. (1984), Perl and Daskin (1985) and Salhi and Rand (1986)). These three subproblems are closely interrelated and cannot be optimized separately without running the risk of arriving at a suboptimal solution. Comprehensive mathematical programming formulations which incorporate simultaneously all aspects of the problem will, in general, contain too many variables and constraints to be easily solved using an exact algorithm. Examples of such formulations and algorithms are presented in sections 5, 6 and 7, but these constitute exceptions rather than the rule. Most practical problems call for a suboptimal algorithm.

Not surprisingly, heuristic algorithms for LRPs exploit the decomposition of the problem into its three components. A typical algorithm will consist of a sequence of routines designed to solve well-known problems such as location-allocation (Cooper, 1963, 1964), the p-median problem (Hakimi, 1964), clustering (Anderberg, 1973) and the VRP (Bodin et al., 1983).

These steps may be executed in different orders, optimally or not, with or without feedback loops and with various degrees of sophistication. The variety of heuristic algorithms is further amplified by the fact that the problem at hand may vary significantly from one author to the next.

In what follows, we attempt to classify, within a unified framework, the various heuristics used in location-routing. Several simplifications are necessary and our description will therefore fail to reflect the level of sophistication of some of the methods. We will denote by the letters **L**, **A** and **R** the location, allocation and routing steps of the various algorithms. If letters are separated by a hyphen, the corresponding steps are executed sequentially. Letters in brackets correspond to simultaneous steps. (Examples : L-A-R; (AR)-L).

Basically, all algorithms described are of two types : location-allocation-routing (LAR) and routing-allocation-location (usually (AR)-L), with various combination of steps being performed sequentially or simultaneously, with possible omission or repetition of some steps, and with the eventual inclusion of feedback loops. **LAR** methods are easily described : facilities are located, users are then allocated to facilities and routes are finally defined. In **ARL** algorithms, sets of routes are first constructed, assuming all facilities "open"; locations are then selected by dropping various facilities from the system and by updating the location and routing decisions. As noted by Feldman et al. (1966), ARL algorithms will be preferred in tightly constrained problems where it is difficult to

construct feasible routes if only a limited number of facilities are open.

Computational comparisons between the various algorithms presented in this section are difficult to make, partly because of the variety of problems treated by the different authors and partly because of insufficient testing or lack of supporting data. The numerical assessment of the various algorithms whose description follows will therefore be somewhat fragmented and incomplete.

3.2 LAR algorithms

Jacobsen and Madsen (1980) presented three different algorithms for the newspaper delivery problem. Two of these belong to the LAR category. The first one can be described as an (LAR) procedure : the location, allocation and routing steps are executed simultaneously in a single pass. The algorithm is described as a "tree-tour heuristic" : a spanning tree satisfying the time and capacity constraint is constructed from the primary facility (printing plant). Secondary facilities are then automatically located at the first stop of each branch. The method is unlikely to produce a very good solution, but is appropriate in the case where there are several arcs to be considered (a few millions in this problem). The second algorithm is representative of several heuristics described in the literature. It is an (LA)-R algorithm in which the alternate location-allocation phase is performed according to the procedure suggested by Rapp (1962) and Cooper (1964) : this method solves the LA problem by alternating between opening a set of facilities and allocating users to the opened facilities. One of its interesting features is that (i) if the optimal locations are known, the optimal allocations are determined by assigning each user to the closest facility; (ii) if the optimal allocations are known, the optimal locations are obtained by solving a series of Weber problems (see for example Kuhn and Kuenne, 1962). For the routing

phase, Jacobsen and Madsen use the well-known Clarke and Wright (1964) algorithm. This method is used for the construction of primary and secondary tours.

Other methods belonging to the (LA)-R category include those of :

- Von Bednar and Strohmeier (1979) : here the location-allocation component is solved by means of a clustering algorithm (Anderberg, 1973) and the ROOT vehicle scheduling package is used for the second component.
- Or and Pierskalla (1979) : the location-allocation phase is first solved by assigning each hospital to the closest blood bank (implicitly determining which facilities to open) and this initial solution is gradually improved by performing exchanges. For the routing phase, the authors consider two types of problems : (i) unconstrained problems : here a series of TSPs are solved by means of a convex hull algorithm (Or, 1976; Or and Pierskalla, 1976); (ii) constrained problems : a series of VRPs are solved by using the Gillett and Miller (1974) algorithm. Extensive computational results using this procedure are reported in the Or and Pierskalla paper and in particular, in Or's thesis (1976). The method appears to have been very well suited to the problem at hand and was applied to real data.
- McLain et al. (1984) : an algorithm developed by Ostresh (1973) is used to determine initial locations and then, the Kuenne and Soland (1972) alternate location-allocation procedure is applied. For the routing phase, the authors solve a multiple TSP using the algorithm proposed by Golden et al. (1977).

In some LAR algorithms, the three steps are executed sequentially (L-A-R). This is the approach used by Watson-Gandy and Dohrn (1973) : an initial choice of depots is made by trial and error, then customers are allocated to their nearest depot and routing costs are evaluated by means

of formula (1) (however, the routes are not determined). The procedure is repeated with a new selection of depots until a satisfactory solution has been reached.

Nambiar et al. (1981) have developed two L-R algorithms (there is no allocation phase since all users are allocated to the same facility) for the rubber factory LRP. In the first case, temporary clusters of users (CSs) are defined so that all elements of the same cluster can be served by the same vehicle without exceeding time or capacity constraints. For every CRPF potential site and for every cluster, the cost of the optimal TSP tour through the CRPF and all CSs of the cluster is computed. This determines an optimal CRPF location. From this location, a VRP through all CSs is solved by means of the Clarke and Wright algorithm. In the second algorithm, temporary clusters are defined and TSPs are solved as above. Then a modified version of Erlenkotter's (1978) simple plant location routine (DUALOC) is used to determine the optimal CRPF locations. Finally, the routing step is performed as in the first algorithm.

The last algorithm described in this subsection is that of Labbé and Laporte (1986) for the post box location problem. The algorithm can best be viewed as an (LA)-(LAR) method. Users are divided into zones and at least ν_k post boxes must be located within zone k. The first step (LA) consists of solving for each zone a ν_k-median problem; let J be the set of selected sites. In the second step (LAR), a sequence of GTSPs are solved optimally. In the objective function, all variables corresponding to nodes initially have a zero weight. At the end of this step, let I be the set of selected sites. If I = J or if cycling occurs, stop. Otherwise, sites belonging to the GTSP solution but not to the ν_k-median solutions are weighted more heavily and the (LAR) step is repeated.

3.3 ARL algorithms

We will describe three algorithms in this section. The first one was proposed by Jacobsen and Madsen (1980) for the newspaper problem. It can be decomposed into two parts : (i) construction of the secondary tours : here the method used belongs to the (AR)-L category. From the primary plant, form secondary tours covering all users. The location of transfer points (secondary facilities) is at this stage considered to be the first stop of each tour. Then reduce the number of transfer points (thus solving the location problem) by means of the DROP procedure of Feldman et al. (1966); (ii) construction of the primary tours : this is a pure routing problem which is solved by the Clarke and Wright algorithm. According the authors, this ARL algorithm produced results comparable with those of the second LAR algorithm presented in section 3.2. Both these algorithms outperformed the tree-tour heuristic.

The Srikar and Srivastava (1984) method belongs to the A-R-L class. Prior to the allocation phase, savings are computed for each user-facility pair; these are the Clarke and Wright savings updated by Tillman (1969) for the multiple facility case. Assign users to facilities according to the highest frequency of positive savings of users across all depots. Solve a routing problem by means of the Clarke and Wright algorithm. Close the depot which would yield the largest savings. Repeat the whole procedure until it is impossible to achieve any further savings. This algorithm appears to have been quite successful : on problems for which the optimum was known, the solution value produced by the heuristic exceeded the global optimum by an average of 4.33% in one-depot problems and 4.68% in two-depot problems.

Finally, the Perl and Daskin (1985) algorithm can be best described as (AR)-(AL)-(AR). The first step is based on the assumption that the way users are grouped into routes is not very much affected by the particular

location of facilities : all facilities are initially opened and a "multiple depot vehicle dispatch problem" is solved as in Golden et al. (1977). In the second step, facility sites are selected and the routes constructed in the first step assigned to them, according to the procedure described in Perl (1987). In the third and last step, the authors solve a "multiple-depot routing-allocation problem", assuming known locations. The exact mathematical programming formulations of the three subproblems are provided in Perl and Daskin. A detailed step by step test of the method was conducted on a problem with 55 users and 15 potential facility sites. It took less than 250 seconds to reach a solution on a CDC Cyber 170/730. Most of this time was spent in the (AL) phase.

4. EXACT ALGORITHMS FOR THE VEHICLE ROUTING PROBLEM

Relatively few exact algorithms have been developed for LRPs and most have been published over the last six or seven years. These are usually based on ILP formulations for the VRP, to which has been added a location dimension. For this reason, we find it useful to briefly review in this section the most frequently used exact algorithms for the VRP. We will mention some VRP algorithms which do not yet possess a counterpart in location-routing, hoping that their inclusion will inspire researchers to develop new types of exact algorithms for LRPs

4.1 Classification

According to a recent survey (Laporte and Nobert, 1987), exact algorithms for the VRP fall into three main categories :

i) direct tree search algorithms;

ii) dynamic programming algorithms;

iii) ILP algorithms.

The latter category can be subdivided, as in Magnanti (1981), into

iiia) set partitioning algorithms;

iiib) vehicle flow algorithms;

iiic) commodity flow algorithms.

4.2 Direct search algorithms

Direct tree search algorithms consist of sequentially building vehicle routes by means of a branch and bound tree. Two basic strategies are used within this category : (i) branching on arcs (a branch corresponds to the inclusion or the exclusion of an arc) - an easy example of this type of branching can be found in Christofides and Eilon (1969); (ii) branching on routes (here, a branch corresponds to a feasible vehicle route) - the Christofides (1976) algorithm is of this type. Christofides et al. (1981a) provide improvements on these early tree search algorithms: these are based on the derivation of sharp lower bounds on the optimum (computed from k-degree center trees and q-routes), in conjunction with Lagrangean relaxation.

4.3 Dynamic programming

Dynamic programming (DP) has been applied to the solution of VRPs by a variety of authors. Eilon et al. (1971) provide a general formulation. Consider a VRP with a fixed number $\bar{m}$ of vehicles, a set $N=\{1,\ldots n\}$ of points (users or cities) and a depot located at city 1. Let $c(S)$ denote the cost of the optimal single route through the depot and all the users of a subset S of $N-\{1\}$. We wish to minimize

$$z = \sum_{j=1}^{\bar{m}} c(S_j) \qquad (3)$$

over all feasible partitions $\{S_1,\ldots,S_{\bar{m}}\}$ of $N-\{1\}$. Let $f_k(U)$ be the minimum cost achievable using k vehicles and delivering to a subset U of $N-\{1\}$. Then, the minimum can be determined through the following recursion formula :

$$f_k(U) = \begin{cases} c(U) & (k=1) \\ \min\limits_{U^* \subseteq U \subseteq N-\{1\}} [f_{k-1}(U-U^*) + c(U^*)] & (k>1) \end{cases} \qquad (4)$$

The solution cost is equal to $f_{\bar{m}}(N-\{1\})$ and the optimal partition corresponds to the optimizing subsets U^* in (4).

Unfortunately, this formulation is in general of little use because of the very large number of states it involves. However, the performance of DP algorithms can be considerably improved by reducing the number of states through state-space relaxation techniques (Christofides et al., 1981b) or by the elimination of dominated and infeasible states (Desrosiers et al., 1986a).

4.4 Set partitioning formulations

ILP algorithms constitute by far the most widely used solution tool for VRPs. Among these, set partitioning formulations cover a wide range of problems. Balinski and Quandt (1964) provide one of the earliest examples of this approach. Consider all feasible routes j and let a_{ij} be a 0-1 coefficient taking the value 1 if and only if city i appears on route j. Let c_j^* be the optimal cost of route j and x_j be a 0-1 variable equal to 1 if and only if route j is used in the optimal solution. Then the problem is to

$$\text{(SP)} \qquad \text{Minimize} \sum_j c_j^* x_j$$

subject to

$$\sum_j a_{ij} x_j = 1 \qquad (i \in N-\{1\}) \qquad (5)$$

$$x_j = 0,1 \qquad (\text{all } j) \qquad (6)$$

However, such formulations suffer the same drawback as the general DP formulation : the number of their variables, even for relatively small problems, is in most cases astronomical. Moreover, the determination of the c_j^*'s usually requires the optimization of a difficult problem such as the TSP. These difficulties can be overcome by using column generation techniques and linear lower bounds on the c_j^*'s. Successful applications of one or both of these techniques can be found in Desrosiers et al. (1984) and in Agarwal (1985).

4.5 Flow formulations

Commodity flow and vehicle flow models constitute the other two major types of ILP formulations. In the first case, the models consider explicitly the quantity of goods travelling on every route and transiting through every facility. These models generally include upper bounds on the flow of goods which can travel through any part of the system, as well as volume dependent processing and transportation costs. Classical examples of such models are the capacitated plant location problem (Kuehn and Hamburger, 1962), the transportation problem and the transshipment problem. Vehicle flow models, on the other hand, deal with the optimal circulation of vehicles and users in the system and do not include costs and constraints directly related to the actual flow of goods. The TSP and the p-median problem fall into this category.

Some problems, such as the capacitated VRP are hybrid cases. They have been modelled as vehicle flow problems by some authors (Laporte and Nobert, 1983a) and as commodity flow problems by others (Garvin et al., 1957; Gavish and Graves, 1979, 1982). Gavish (1984) has proposed various vehicle flow and commodity flow formulations for the capacitated VRP. His results (reported in Laporte and Nobert (1987)) include comparisons of the lower bounds provided by solving the linear relaxation of various models. These follow a similar analysis conducted by Wong (1980) on a number of ILP formulations for the TSP.

It is often convenient to classify VRP formulations according to the number of indices of the flow variables. Common cases are :

i) $x_{ij} = \begin{cases} 1 \text{ if a vehicle travels on arc } (i,j) \\ 0 \text{ otherwise} \end{cases}$

ii) $x_{ijk} = \begin{cases} 1 \text{ if vehicle k travels on arc } (i,j) \\ 0 \text{ otherwise} \end{cases}$

iii) w_{ij} = quantity of goods travelling on arc (i,j)

iv) $w_{ij\ell}$ = quantity of goods travelling on arc (i,j) and destined for city ℓ.

v) $z_{ik} = \begin{cases} 1 \text{ if city i is served by vehicle k} \\ 0 \text{ otherwise} \end{cases}$

Moreover, depending whether the distance (or travel time) matrix of the problem under study is symmetrical or not, (i,j) will represent an undirected arc (edge) or a directed arc. In symmetrical problems, it is usually unnecessary to define variables for $i > j$.

There are certain advantages and disadvantages in using two-index or three-index variables in VRP or for that matter, LRP formulations. Two-index formulations are concise and involve a relatively small number of variables (particularly in the symmetrical case). However, they cannot take into account different vehicle costs and characteristics and can therefore only be applied under the assumption that the vehicle fleet is homogeneous. Successful implementations of such formulations (using x_{ij} variables) are reported by Fleischmann (1982) and Laporte et al. (1985), for example. Three-index formulations are more versatile but more costly. Classical examples of such formulations are the vehicle flow models proposed by Golden et al. (1977) (using x_{ijk} variables) and by Fisher and Jaikumar (1978) (using x_{ijk} and z_{ik} variables). However, to the author's knowledge, three-index formulations have never led to the successful implementation of an **exact** algorithm for the VRP. In the next three sections, we examine some three-index and two-index flow formulations for LRPs. In the latter case, successful exact algorithms are also described.

5. THREE-INDEX VEHICLE FLOW FORMULATIONS

Three-index vehicle flow formulations appear to be the most widely used in the location-routing literature. Since they are very flexible, they facilitate the inclusion of a variety of constraints and the definition of very general objective functions. However, their size renders

their treatment by exact methods impractical. Four papers (Golden et al., 1977; Or and Pierskalla, 1979; Srikar and Srivastava, 1983; Perl and Daskin, 1985) propose such formulations. The first three papers consider an M(T) problem, with depots to be located at the first layer and users at known locations at the second layer. The Perl and Daskin article treats the "classical" M(R,T) problem, with known supply sources at the first layer, and the next two layers as in the M(T) model. These formulations all rest upon the same principle and differ mainly in the number of cost components and restrictions they incorporate. If we remove from these formulations elements which are directly related to some particular characteristics of the problem at hand, the similarities become even more striking. By disregarding the secondary local features contained in the formulations, and by making minor adjustments to the Perl and Daskin model, we arrive at a fairly comprehensive formulation which includes as special cases, all four models considered.

5.1 Objective, constraints and notation

The classical M(R,T) LRP consists of simultaneously

i) selecting facility sites at the first layer;

ii) determining the composition of the vehicle fleet based at each facility;

iii) constructing optimal delivery routes from the supply sources to the facilities located at the second layer and from these facilities to users;

The solution must be such that

iv) the total system cost is minimized;

v) all user requirements are satisfied without exceeding vehicle capacities;

vi) the number of vehicles used does not exceed a given bound;

vii) route lengths are within a given maximum distance;

viii) route durations do not go beyond a preset time;

ix) vehicle routes usually pass through only one facility;

x) facility throughput capacities are respected.

In order to formally express these constraints and objective, the following notation is introduced. Define

Sets

L : the set of all supply sources (first layer)

I : the set of all potential facility sites (second layer)

J : the set of all users (third layer)
(Note : I, J and L are pairwise disjoint.)

K : the set of all vehicles.

Parameters

$\bar{m} = |K|$: the maximum allowed number of vehicles

$C = (c_{ij})$ $(i,j \in I \cup J)$: a distance matrix (undefined if $i,j \in I$)

$T = (t_{ijk})$ $(i,j \in I \cup J;\ k \in K)$: a three-dimensional travel time array for all vehicles and for all pairs of facilities and users

g_i : the fixed cost of establishing facility $i \in I$

v_i : the variable cost per throughput unit at facility $i \in I$

V_i : the maximum throughput at facility $i \in I$

$s_{\ell i}$: the unit trunking cost from $\ell \in L$ to $i \in I$

d_j : the requirement of user $j \in J$

τ_{kj} : the time required by vehicle k to unload at user $j \in J$

D_k : the capacity of vehicle (or route) k

E_k : the maximum allowable length (in distance units) of route k

T_k : the maximum allowed duration of route k

p_k : the cost per distance unit of delivery vehicle on route k

q_{ij} : a fixed cost incurred for delivering from facility $i \in I$ to user $j \in J$.

Variables

$$x_{ijk} = \begin{cases} 1 & \text{if } i \text{ immediately precedes } j \text{ on route } k \ (i,j \in IUJ;\ k \in K) \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if site } i \in I \text{ is used to locate a facility} \\ 0 & \text{otherwise} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if user } j \in J \text{ is served from facility } i \in I \\ 0 & \text{otherwise} \end{cases}$$

$w_{\ell i}$ = quantity of goods shipped from supply source $\ell \in L$ to facility $i \in I$.

5.2 Formulation

The problem formulation is as follows :

(LRP1) Minimize $$\sum_{i \in I} g_i y_i + \sum_{\ell \in L} \sum_{i \in I} s_{\ell i} w_{\ell i} + \sum_{i \in I} \sum_{j \in J} (v_i d_j + q_{ij}) z_{ij} + \sum_{k \in K} \sum_{i \in IUJ} \sum_{j \in IUJ} p_k c_{ij} x_{ijk}$$

subject to

$$\sum_{k \in K} \sum_{i \in IUJ} x_{ijk} = 1 \qquad (j \in J) \qquad (7)$$

$$\sum_{j \in J} d_j \sum_{i \in IUJ} x_{ijk} \leq D_k \qquad (k \in K) \qquad (8)$$

$$\sum_{i \in IUJ} \sum_{j \in IUJ} c_{ij} x_{ijk} \leq E_k \qquad (k \in K) \qquad (9)$$

$$\sum_{j \in J} \tau_{kj} \sum_{i \in IUJ} x_{ijk} + \sum_{i \in IUJ} \sum_{j \in IUJ} t_{ijk} x_{ijk} \leq T_k \qquad (k \in K) \qquad (10)$$

$$\sum_{k \in K} \sum_{i \in S} \sum_{j \in (IUJ)-S} x_{ijk} \geq 1 \qquad (2 \leq |S| \leq |IUJ|;\ S \subseteq IUJ\ ;\ S \cap J \neq \emptyset) \qquad (11)$$

$$\sum_{j \in IUJ} x_{ijk} - \sum_{j \in IUJ} x_{jik} = 0 \qquad (k \in K,\ i \in IUJ) \qquad (12)$$

$$\sum_{i \in I} \sum_{j \in J} x_{ijk} \leq 1 \qquad (k \in K) \qquad (13)$$

$$\sum_{\ell \in L} w_{\ell i} - \sum_{j \in J} d_j z_{ij} = 0 \qquad (i \in I) \qquad (14)$$

$$\sum_{\ell \in L} w_{\ell i} \leq V_i y_i \qquad (i \in I) \qquad (15)$$

$$-z_{ij} + \sum_{s \in I \cup J} (x_{isk} + x_{sjk}) \leq 1 \qquad (i \in I,\ j \in J,\ k \in K) \qquad (16)$$

$$x_{ijk} = 0,1 \qquad (i \in I,\ j \in J,\ k \in K) \qquad (17)$$

$$y_i = 0,1 \qquad (i \in I) \qquad (18)$$

$$z_{ij} = 0,1 \qquad (i \in I,\ j \in J) \qquad (19)$$

$$w_{\ell i} \geq 0 \qquad (\ell \in L,\ i \in I) \qquad (20)$$

In this formulation, the objective function is the sum of facility fixed costs, trunking costs, variable warehousing costs and delivery costs, respectively. Constraints (7) ensure that every user belongs to one and only one route. Constraints (8), (9) and (10) guarantee that vehicle capacities, maximum route lengths and maximum route durations, respectively, are respected. Constraints (11) are connectivity constraints : they ensure that every user is on a route connected to the set of facilities. Constraints (12) are flow conservation equations : any point of IUJ must be entered and left by the same vehicle. Constraints (13) stipulate that a vehicle can depart only once from a facility. This prevent cases where a vehicle leaves the same facility for two different users and also passes through more than one facility. Constraints (14) ensure that the flow entering a facility is equal to the flow out of that facility. Constraints (15) limit the flow through a facility to the capacity of that facility. Constraints (16) specify that y_{ij} must be equal to 1 if facility i and user j belong to the same route k. Finally, constraints (17) to (20) impose bounds and integrality conditions on the variables.

5.3 Special cases

As mentioned earlier, the formulation just presented contains (with minor simplifications) the main models described in the literature. The Perl and Daskin formulation is very close to (LRP1) : it suffices to remove constraints (10) and to set the q_{ij}'s equal to zero. The other three papers considered (Golden et al., 1977; Or and Pierskalla, 1979; Srikar and Srivastava, 1983) describe two-layer problems. Therefore, in all three cases, we can set $L = \emptyset$, eliminate the corresponding variables ($w_{\ell i}$) and parameters ($s_{\ell i}$) as well as constraints (14), (15) and (20). Apart from the fact that these three papers possess slightly different objective functions, (all special cases of that used in (LRP1)), they contain the following differences :

i) Golden et al. do not consider maximum distance restrictions (9), pairing variables y_{ij}, nor constraints (16).

ii) Or and Pierskalla do not include time limits (10) and omit constraints (13) (thus allowing routes through several facilities and in turn, because of constraints (16), possibly assigning a user to more than one facility).

iii) Srikar and Srivastava use, instead of constraints (11), the subtour elimination constraints of Miller et al. (1960); they do not define variables y_{ij}, nor constraints (16). Finally, they specify the number of facilities which are to be opened.

6. TWO-INDEX VEHICLE FLOW FORMULATIONS AND ALGORITHMS FOR SYMMETRICAL PROBLEMS

Exact algorithms for symmetrical TSPs and VRPs (i.e. problems having a symmetrical distance matrix) are traditionally based on ILP models and algorithms, while network flow approaches are more often used for asymmetrical problems (see for example Lawler et al. (1985); Laporte

and Nobert (1987)). These models are usually derived from the classical TSP formulation first proposed by Dantzig et al. (1954). Not surprisingly, models for symmetrical LRPs will also belong to the same class. We present in this section three classes of LRP formulations and algorithms partially designed by the author. We show how they relate to some well-known TSP and VRP models. For the sake of brevity, we restrict ourselves to two-layer problems, omitting other multi-layer problems and various classes of LRPs such as the GTSP (an exact algorithm for symmetrical GTPSs is described in Laporte and Nobert (1983b)).

6.1 A classical TSP formulation and some extensions

The first ILP formulation for the TSP is attributed to Dantzig et al. (1954).

Sets

$N = \{1,\ldots,n\}$ a set of nodes (points, customers or cities)

$\bar{S} = N-S$

Parameters

$C = (c_{ij})$ $(i,j \in N)$, a symmetrical distance matrix

Variables

x_{ij} : a 0-1 variable equal to 1 if and only if edge (i,j) belongs to the optimal solution. x_{ij} need only be defined if $i<j$. Here and elsewhere, x_{ij} must be interpreted as x_{ij} whenever $i > j$.

The problem is then to

(TSP) Minimize $\sum_{i,j \in N} c_{ij} x_{ij}$

subject to

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (k \in N) \qquad (21)$$

$$\sum_{i,j \in S} x_{ij} \leq |S|-1 \qquad (S \subset N;\ 2<|S|<n-2) \qquad (22)$$

$$x_{ij} = 0,1 \qquad (i,j \in N) \qquad (23)$$

In this formulation, constraints (21) are **degree constraints** : they specify the degree of each node. Constraints (22) are **subtour elimination constraints** : they prevent the formation of subtours over proper subsets of N. Constraints (23) are **integrality constraints.**

It is well known (Dantzig et al., 1954) that, if (21) holds, constraints (22) are equivalent to the following **connectivity constraints** :

$$\sum_{\substack{i \in S, j \in \bar{S} \\ \text{or } i \in \bar{S}, j \in S}} x_{ij} \geq 2 \qquad (S \subset N;\ 2<|S|<n-2) \qquad (24)$$

When, as in (TSP) for example, the degree of every node of S is equal to 2, this equivalence can be extended to :

$$\sum_{i,j \in S} x_{ij} \leq |S|-r \iff \sum_{\substack{i \in S, j \in \bar{S} \\ \text{or } i \in \bar{S}, j \in S}} x_{ij} \geq 2r \quad (S \subset N;\ r \in |R|) \qquad (25)$$

By selecting an appropriate value of r, it is possible to use either of the two constraints defined by (25) to solve a large class of VRPs (see the work of Nobert (1982), Laporte and Nobert (1983a) and Laporte et al. (1984a, 1985)).

Now define $\Delta(S)$, the **degree of S** :

$$\Delta(S) = \sum_{i \in S} \left(\sum_{k>i} x_{ik} + \sum_{k<i} x_{ki} \right) \qquad (26)$$

It can be shown (Laporte, 1986) that when $\Delta(S) \neq 2|S|$, (25) no longer holds. However the following equivalence is still valid :

$$\sum_{i,j\in S} x_{ij} \leq \frac{1}{2}\Delta(S)-r \iff \sum_{\substack{i\in S, j\in \bar{S} \\ \text{or } i\in\bar{S}, j\in S}} x_{ij} \geq 2r \quad (S\subset N; 2\leq|S|\leq n-2; r\in|R) \qquad (27)$$

We will now use this result to formulate a class of LRPs.

6.2 The single depot LRP

In this very simple form of the LRP, we consider a set N of users and a subset $I\subseteq N$ of potential sites for a single facility, in order to minimize the total routing cost for a fleet of exactly m vehicles based at that facility. (We deviate here from the standard LRP framework presented in section 1.3 since now, the facility set and the user set are not disjoint).

Define a binary variable y_i equal to 1 if and only if the facility is located at node i. The remaining notation is as in (TSP). The problem is then to

(LRP2) Minimize $\sum_{i,j\in N} c_{ij}x_{ij}$

subject to

$$\sum_{i\in I} y_i = 1 \qquad (28)$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2+2(m-1)y_i \qquad (i\in I) \qquad (29)$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (i\in N-I) \qquad (30)$$

$$\sum_{i,j\in S} x_{ij} \leq \frac{1}{2}\Delta(S)-1 \qquad (S\subset N;\ 2\leq|S|\leq n-2) \qquad (31)$$

$$y_i = 0,1 \qquad (i\in I) \qquad (32)$$

$$x_{ij} = \begin{cases} 0,\ 1,\ 2 & (i \text{ or } j\in I) \\ 0,1 & (i,j\in N-I) \end{cases} \qquad (33)$$

Most constraints of this model are self-explanatory. Constraints (31) are derived as follows : any feasible solution must correspond to a connected graph, therefore the following constraint must hold :

$$\sum_{i\in S, j\in \bar{S}} x_{ij} \geq 2 \qquad (S\subset N;\ 2\leq |S| \leq n-2) \qquad (34)$$

$$\text{or} \quad i\in \bar{S}, j\in S$$

By equivalence (27), (34) can be rewritten as (31). Note that it would not have been valid to use (22) instead of (31), since $\Delta(S) > 2$ whenever $\sum_{i\in S} y_i = 1$.

Laporte (1986) shows that (LRP2) is equivalent to the model presented in Laporte and Nobert (1981) for the same problem, and how it can be extended to the case where m is a variable bounded above by $\bar{m}$.

In Laporte and Nobert (1981), (LRP2) is solved by means of a constraint relaxation method inspired by the branch and bound algorithm for the TSP devised by Miliotis (1976). (LRP2) is solved by initially relaxing the subtour elimination constraints (31) and the integrality conditions on the variables (but by retaining their upper bounds). Integrality is gradually regained by branch and bound; at an integer solution, a check for violated subtour elimination constraints is made, the constraint corresponding to the subtour involving the least number of nodes is introduced and the problem is reoptimized. The authors have experimented with two types of distance matrices (satisfying the triangle inequality, or not) and two branching strategies. In all problems, I was equal to N. Problems involving up to 50 nodes were solved optimally within 900 seconds on the University of Montreal Cyber 173 computer.

6.3 A family of multi-depot LRPs

Laporte et al. (1983) studied a variant of (LRP2) in which P facilities, instead of only one, have to be located. The triangle inequality is assumed to be satisfied in I, i.e.

$$c_{ki} + c_{ij} > c_{kj} \qquad (i \in I) \qquad (35)$$

Under this condition, it can be shown that there will be only one vehicle per opened facility in the optimal solution. As in (LRP1), g_i was defined as the cost of opening a facility at site i. The problem can be formulated as (LRP2) with m=1, a right-hand side of P in (28) and a new objective :

(LRP3) Minimize $\sum_{i,j \in N} c_{ij} x_{ij} + \sum_{i \in I} g_i y_i$.

Two variants of (LRP3) were considered :

i) $g_i > 0$ for all $i \in I$ and $P = |I|$;

ii) $g_i = 0$ for all $i \in I$ and $P < |I|$.

Test problems were generated and solved by adapting Miliotis' (1978) REVERSE algorithm for the TSP :

Step 1. Solve (LRP3) by relaxing constraints (31) and the integrality conditions on the variables.

Step 2. Identify violated subtour elimination constraints (even if the solution at hand is not necessarily integer) and introduce one such constraint for each illegal component (i.e. a set of nodes disjoint from I and connected by "positive edges"). Reoptimize until no illegal component remains.

Step 3. Reach integrality by gradually introducing Gomory cuts. Stop if a feasible solution has been reached. Otherwise, proceed to step 2.

The algorithm was constructed by extending or otherwise modifying the Land-Powell (1973) routines adapted by Miliotis (1978) for integer arithmetic. For each of the two cases considered, problems containing up to 40 nodes were solved to optimality.

A simple variant of (LRP3) occurs when $g_i = 0$ for all $i \in I$ and $P = |I|$. Then the problem (LRP4) consists of "covering" the nodes with a set of Hamiltonian cycles, each containing at least one node belonging to

I. (LRP4) is in fact a relaxation of (TSP) in which some subtours are allowed. Its formulation is that of (TSP) with the exception that constraints (22) are replaced by

$$\sum_{i,j \in S} x_{ij} \leq |S|-1 \qquad (S \subset N;\ S \cap I = \emptyset;\ 2 \leq |S| \leq n-2) \qquad (36)$$

It should be noted that (LRP4) is not, properly speaking, an LRP; the facilities having no fixed costs, their precise locations on the various Hamiltonian cycles contained in the solution are immaterial, as long as they correspond to elements of I.

The problem was solved by means of the REVERSE algorithm for various combinations of $|N|$ and $|I|$ and, as expected, computation times were very good. For example, all problems with $|N| = 50$ were solved within 45 seconds on a CYBER 173.

In a recent paper, Branco and Coelho (1986) proposed a set partitioning formulation as well as four heuristics for (LRP4). The quality of these heuristics is difficult to assess since no comparisons with the global optima are made.

Finally, Laporte et al. (1983) also studied two LRPs belonging to the same family. In the first of these, condition (35) is relaxed, ans it is assumed that if a facility is opened at site i, then exactly m_i vehicles will be based at that facility. The problem appears to be considerably more difficult than (LRP3). In the second case, it is assumed that C does not satisfy the triangle inequality and that multiple passages through the same node are allowed. The problem is relatively easy to solve; it constitutes in fact a relaxation of the problem which consists of determining the shortest complete cycle in a graph, previously analyzed by Miliotis et al. (1981) and recently reexamined by Jonker (1986) and by Volgenant et al. (1986).

6.4 A multi-depot capacitated LRP

We now revert to our original framework (section 1.3) and consider an M(T) problem in which I, the set of potential facilities, is disjoint from J, the set of users. Laporte et al. (1986c) have studied such a problem belonging to this class and involving (i) simultaneous location and routing, (ii) user demands and capacity constraints, (iii) fixed costs in facilities and vehicles, (iv) bounds on the number of opened facilities, (v) bounds on the number of vehicles per facility. One interest of the paper is that it describes a procedure for generating cuts in order to eliminate solutions in which some vehicles start their journey at a given facility and terminate it at a different facility. These cuts known as **chain barring constraints** are also used in other types of multi-depot VRPs (Laporte et al., 1984c).

Let I, J, d_j, g_i, y_i be defined as in (LRP1), section 5.1. Here and in the remainder of this paper, C is a matrix of travel **costs**. We also use the following notation :

Parameters

D : the capacity of a vehicle

P : the number of facilities in the optimal solution. P lies between two prespecified bounds $\underline{P} > 1$ and $\bar{P} < |I|$.

Function

$\lceil x \rceil$: the smallest integer greater than or equal to x if $x > 0$; 1 otherwise.

Variables

m_i : the number of vehicles based at facility i; m_i lies between two bounds : $1 < \underline{m}_i < m_i < \bar{m}_i$.

x_{ij} : the number of times (0, 1 or 2) edge (i,j) is used in the optimal solution; x_{ij} is undefined if $i > j$, if $i,j \in I$ or if $d_i + d_j > D$.

The problem consists of selecting facility sites (when $P < |I|$), of determining how many vehicles are based at each opened facility and of establishing vehicle routes in such a way that

i) each route starts and ends at the same facility;

ii) all of the service requirement of a user is met by only one vehicle (the same user may be visited more than once if this saves distance, but only one of the vehicle visits is used to meet the user's requirement);

iii) the sum of all requirements satisfied by any vehicle does not exceed D;

iv) the numbers of facilities and of vehicles per facility lie within their respective bounds;

v) the total system cost is minimized.

The problem can be formulated as follows :

(LRP5) Minimize $\sum_{i,j \in I \cup J} c_{ij} x_{ij} + \sum_{i \in I} (g_i y_i + f_i m_i)$

subject to

$$\sum_{i<k} x_{ik} + \sum_{k<j} x_{kj} = 2 \qquad (k \in J) \qquad (37)$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2m_k \qquad (k \in I) \qquad (38)$$

$$\sum_{i,j \in S} x_{ij} \leq |S| - \left\lceil \frac{\sum_{k \in S} d_k}{D} \right\rceil \qquad (S \subseteq J;\ |S| \geq 3) \qquad (39)$$

$$x_{i_1 i_2} + 3x_{i_2 i_3} + x_{i_3 i_4} \leq 4 \qquad (i_1, i_4 \in I;\ i_2, i_3 \in J) \qquad (40)$$

$$x_{i_1 i_2} + x_{i_{h-1} i_h} + 2 \sum_{i,j \in \{i_2, \ldots i_{h-1}\}} x_{ij} \leq 2h-5 \qquad (41)$$

$$(h \geq 5;\ i_1,\ i_h \in I;\ i_2, \ldots, i_{h-1} \in J)$$

$$y_i \leq m_i \leq M y_i \qquad (i \in I) \qquad (42)$$

$$\underline{m}_i \leq m_i \leq \bar{m}_i \qquad (i \in I) \qquad (43)$$

$$\underline{P} \leq \sum_{i \in I} y_i \leq \bar{P} \qquad (44)$$

$$y_i = 0,1 \qquad (i \in I) \qquad (45)$$

$$x_{ij} = \begin{cases} 0,1 & (i,j \in J) \\ 0,1,2 & (i \text{ or } j \in I) \end{cases} \qquad (46)$$

The objective function and most constraints of this formulation are self-explanatory. We provide details on constraints (39)-(41). Constraints (39) are imposed to eliminate subtours which are either disconnected from I and vehicle routes whose sum of requirements exceeds D. They are derived as follows : any subset S of J must be entered and left by at least $\left\lceil \frac{\sum_{k \in S} d_k}{D} \right\rceil$ vehicles, hence the following constraint is valid :

$$\sum_{\substack{i \in S, j \in \bar{S} \\ \text{or } i \in \bar{S}, j \in S}} x_{ij} \geq 2 \left\lceil \frac{\sum_{k \in S} d_k}{D} \right\rceil \qquad (S \subseteq J;\ |S| \geq 3) \qquad (47)$$

When (37) is satisfied, (47) is equivalent to (39) (see (25)). Constraints (40) and (41) are chain barring constraints : they ensure that each route starts and ends at the same facility. They prohibit solutions containing chains of edges between two facilities i_1, i_h, and h-2 nodes $i_2, \ldots i_{h-1}$ belonging to J. The constraints are derived by considering three cases : $h < 3$, $h = 4$ and $h > 5$. The first case is of no interest since it cannot occur in an optimal solution. For $h = 4$ and $h > 5$, the required constraint is derived as follows. Define

$$X = x_{i_1 i_2}, \quad Y = x_{i_{h-1} i_h} \quad \text{and} \quad Z = \sum_{i,j \in \{i_2, \ldots, i_{h-1}\}} x_{ij} \qquad (48)$$

and let $\bar{X}$, $\bar{Y}$, $\bar{Z}$ be the values taken by X, Y and Z respectively. In a feasible solution to (LRP5), $\bar{X}, \bar{Y} \in \{0,1,2\}$ and $\bar{Z}$ is an integer in $[0, u(\bar{X}, \bar{Y})]$

where $u(\bar{X},\bar{Y})$ is an upper bound on $\bar{Z}$ whose value depends on h, $\bar{X}$ and $\bar{Y}$. Details of the computation of $u(\bar{X},\bar{Y})$ are provided in Laporte et al. (1986c). We seek the strongest constraint of the form

$$aX + bY + Z \leq d \tag{49}$$

which satisfies

$$d - a\bar{X} - b\bar{Y} \geq u(\bar{X},\bar{Y}) \qquad (\bar{X},\bar{Y}\in\{0,1,2\}) \tag{50}$$

and

$$a,\ b\in \mathbb{R};\quad d \geq 0 \tag{51}$$

while cutting off any infeasible solution for which $\bar{X} = \bar{Y} = 1$ and $\bar{Z} = h-3$. The strongest cut is obtained by minimizing (d-a-b), the value taken by the left-hand side of (50) when $\bar{X} = \bar{Y} = 1$, subject to (50) and (51).

It is worth noting that the problem of illegal chains does not occur so acutely in three-index formulations such as (LRP1) since it is then straightforward to impose (constraints (13)) that the same vehicle can leave only one facility. The difficulty in two-index formulations arises from the fact that vehicles are not identified by the variables.

The algorithm proposed for the solution of (LRP5) also belongs to the class of constraint relaxation methods. It is of the same basic type as that used for the solution of (LRP2) : here, the constraints which are initially relaxed are (i) the subtour elimination constraints; (ii) the chain barring constraints and (iii) the integrality constraints. In addition, checks are carried out in the course of the branch and bound process to fix at zero certain variables which could not enter the solution without violating the vehicle capacity constraints. Tests indicate that problems involving up to 20-25 nodes can be solved to optimality within 300 seconds, on the University of Montreal CYBER 855 computer.

7. TWO-INDEX VEHICLE FLOW FORMULATIONS AND ALGORITHMS FOR ASYMMETRICAL PROBLEMS

We present in this section a general methodology for solving asymmetrical LRPs. The fundamental idea consists of transforming the original LRP into a problem containing an easy-to-solve relaxation; constraints not included in the relaxation can be dealt with by a branch and bound procedure. More precisely, we transform the problems into TSPs with specified nodes (Laporte et al., 1984b), under a variety of side constraints. This approach was recently used with success by Laporte et al. (1986b) to solve two classes of LRPs. The proposed methodology will be illustrated on a three-layer problem. By adjusting two parameters, the four scenarios M(R,R), M(R,T), M(T,R), M(T,T) can be covered. The model can easily be adapted to other situations.

7.1 Problem definition

As previously, let L be the set of primary facilities, I, the set of secondary facilities and J, the set of users. These three sets are pairwise disjoint. Assume all elements of J and L have to be included in the solution whereas only a subset of I has to be used; the elements of I represent potential locations. The cost of establishing a facility at site $i \in I$ is equal to g_i. A maximum of $\bar{M}_\ell$ vehicles can be based at primary facility $\ell \in L$; similarly, a maximum of $\bar{m}_i$ vehicles can be based at secondary facility $i \in I$. These vehicles have fixed costs of F and f respectively. Various constraints derived from vehicle capacities, maximum route length restrictions, etc. may be imposed on the routes emanating from any layer. Let $C = (c_{ij})$ $(i \neq j)$ be the travel cost matrix defined on $(L \cup I \cup J)^2$. The problem (LRP6) consists of selecting secondary facility sites, of determining the size of the vehicle fleet at all opened facilities and of establishing vehicle routes in such a way that

i) each route starts and ends at the same facility;

ii) all side constraints on the routes are satisfied;

iii) the number of vehicles based at any facility does not exceed its upper bound;

iv) the total system cost is minimized.

We represent this problem by a directed graph G = (N,A,C) where $N = L \cup I \cup J$, $A = \{(i,j);i,j\in N,i\neq j\}$ and C is the travel cost matrix associated with A.

7.2 Graph extension

Several authors (see for example Lenstra and Rinnooy Kan (1975), Laporte et al. (1986b) have suggested graph transformations to solve a variety of TSP extensions. These transformations are such that every feasible solution to the original problem corresponds to a Hamiltonian circuit on a subgraph of the new graph. Any feasible solution on the transformed graph can be interpreted as a solution on the original graph. In order to achieve the desired transformation for (LRP6), we proceed as follows.

Graph G is transformed into G' = (N',A',C'). The new set of nodes N' is the union of three pairwise disjoint sets of nodes :

i) a set L' consisting of the union of $|L|$ sets of nodes $L'_\ell (\ell=1,\ldots,|L|)$ associated with the primary facilities and such that $|L'_\ell| = \bar{M}_\ell + 1$;

ii) a set I' consisting of the union of $|I|$ sets of nodes $I'_i (i=1,\ldots,|I|)$ associated with the secondary facilities and such that $|I'_i| = \bar{m}_i+1$;

iii) the set J of user nodes.

It is convenient to label the nodes of L' as $\ell_{rs} (r=1,\ldots,|L|;\ s=1,\ldots,\bar{M}_r+1)$ and those of I' as $i_{rs} (r=1,\ldots,|I|;\ s=1,\ldots,\bar{m}_r+1)$. Further, define $N'_1 = L' \cup J \cup \left(\bigcup_{i\in I^*} I'_i \right)$ where I* is the index set of secondary facilities which must be included in the solution, and $N'_2 = N'-N'_1$. All

elements of N_1' are designated as **specified nodes** and those of N_2' as **unspecified nodes.**

The new set of arcs A' can be defined as $A' = \bigcup_{t=1}^{11} A_t$ where

$A_1 = L' \times \{i_{r,1} : r=1,\ldots,|I|\}$ is a set of arcs from primary to secondary facilities;

$A_2 = \{i_{r,\bar{m}_r+1} : r=1,\ldots,|I|\} \times L'$ is a set of arcs from secondary to primary facilities;

$A_3 = I' \times J$ is a set of arcs from secondary facilities to users;

$A_4 = J \times I'$ is a set of arcs from users to secondary facilities;

$A_5 = \bigcup_{r=1}^{|L|} \bigcup_{s=1}^{\bar{M}_r} \{(\ell_{rs}, \ell_{r,s+1})\}$ is a set of intra-primary facility arcs;

$A_6 = \bigcup_{\substack{r,r'=1 \\ r\neq r'}}^{|L|} \{(\ell_{r,\bar{M}_r+1}, \ell_{r',1})\}$ is a set of inter-primary facility arcs;

$A_7 = \bigcup_{r=1}^{|I|} \bigcup_{s=1}^{\bar{m}_r} \{(i_{rs}, i_{r,s+1})\}$ is a set of intra-secondary facility arcs;

$A_8 = \bigcup_{\substack{r,r'=1 \\ r\neq r'}}^{|I|} \{(i_{r,\bar{m}_r+1}, i_{r',1})\}$ is a set of inter-secondary facility arcs;

$A_9 = \{(u,u) : u\in N_2', u=i_{r,1}\}$ and

$A_{10} = \{(u,u) : u\in N_2', u=i_{r,s}, s>1\}$ are two sets of loops associated with unspecified nodes;

$A_{11} = \{(j,j') : j,j'\in J, j\neq j'\}$ is a set of inter-user arcs.

The cost matrix $C' = (c'_{uv})$ associated with A' is then defined by the following formulas :

$$c'_{uv} = \begin{cases} F + c_{uv} & (u,v) \in A_1 \\ f + c_{uv} & (u,v) \in A_3 \\ -g_r & (u,v) = (i_{r,1}, i_{r,1}) \in A_9 \\ \theta_1 c_{uv} & (u,v) \in A_8 \\ \theta_2 c_{uv} & (u,v) \in A_{11} \\ c_{uv} & (u,v) \in A_2 \cup A_4 \\ 0 & (u,v) \in A_5 \cup A_6 \cup A_7 \cup A_{10} \\ \infty & \text{otherwise} \end{cases} \qquad (52)$$

In (52), θ_t (t=1,2) corresponds to the distribution mode used between layer t and layer t+1 :

$$\theta_t = \begin{cases} 1 & \text{if } M_t = T \\ \infty & \text{if } M_t = R \end{cases} \qquad (53)$$

7.3 Interpretation

Before proceeding to the formulation of the problem, it appears useful to illustrate a feasible solution on G' (figure 2) and its translation on G (figure 3). In this example, there are two primary facilities (with a maximum of two vehicles each) and four secondary facilities (with a maximum of three vehicles each). These facilities are represented by rectangles containing as many dots as one plus the maximum number of vehicles. A black dot indicates that the corresponding facility must necessarily appear in the optimal solution. The solution on G' corresponds to a Hamiltonian circuit on a subset of N'. Primary vehicle routes are represented by dashed lines whereas secondary vehicles tours correspond to full lines. The dotted arrows between primary facilities have no particular meaning. Loops shown on secondary facility S_3 mean that this facility is not used. Arrows emanating from a node within a facility and pointing outside the facility correspond to used vehicles; arrows having their two ends within the same facility corrpespond to unused vehicles. Note that

i) vehicle tours start and end at the same facility;

ii) vehicle nodes are only used if the associated facility is opened.

7.4 The model

In order to formulate the problem, first define 0-1 variables x_{ij} :

i) if $i \neq j$ (i.e. $(i,j) \in A' - A_9 - A_{10}$), $x_{ij}=1$ if arc (i,j) is used in the optimal solution and $x_{ij}=0$ otherwise;

ii) if $i=j$ (i.e. $(i,j) \in A_9 \cup A_{10}$), $x_{ij}=0$ if node i of N_2' is used in the optimal solution and $x_{ii}=1$ otherwise.

In what follows, variables x_{ij} which are undefined must be interpreted as 0. The formulation is then :

(LRP6) Minimize $\sum_{i,j \in N'} c'_{ij} x_{ij}$

subject to

$$\sum_{i \in N'} x_{ik} = 1 \qquad (k \in N') \qquad (54)$$

$$\sum_{j \in N'} x_{kj} = 1 \qquad (k \in N') \qquad (55)$$

$$\sum_{i,j \in S} x_{ij} \leq |S|-1 \qquad (S \subset N, |S| \geq 2, \text{ i or } j \in N_1') \qquad (56)$$

If a route (defined on G) starts at a given facility, it must end at the same facility. (57)

Routes (defined on G) must satisfy all side conditions. (58)

$$x_{uu} \leq x_{vv} \qquad (u = i_{r,1} \text{ for some } r \text{ and } v = i_{rs} \text{ where } s > 1) \qquad (59)$$

$$x_{ij} = 0,1 \qquad (i,j \in N') \qquad (60)$$

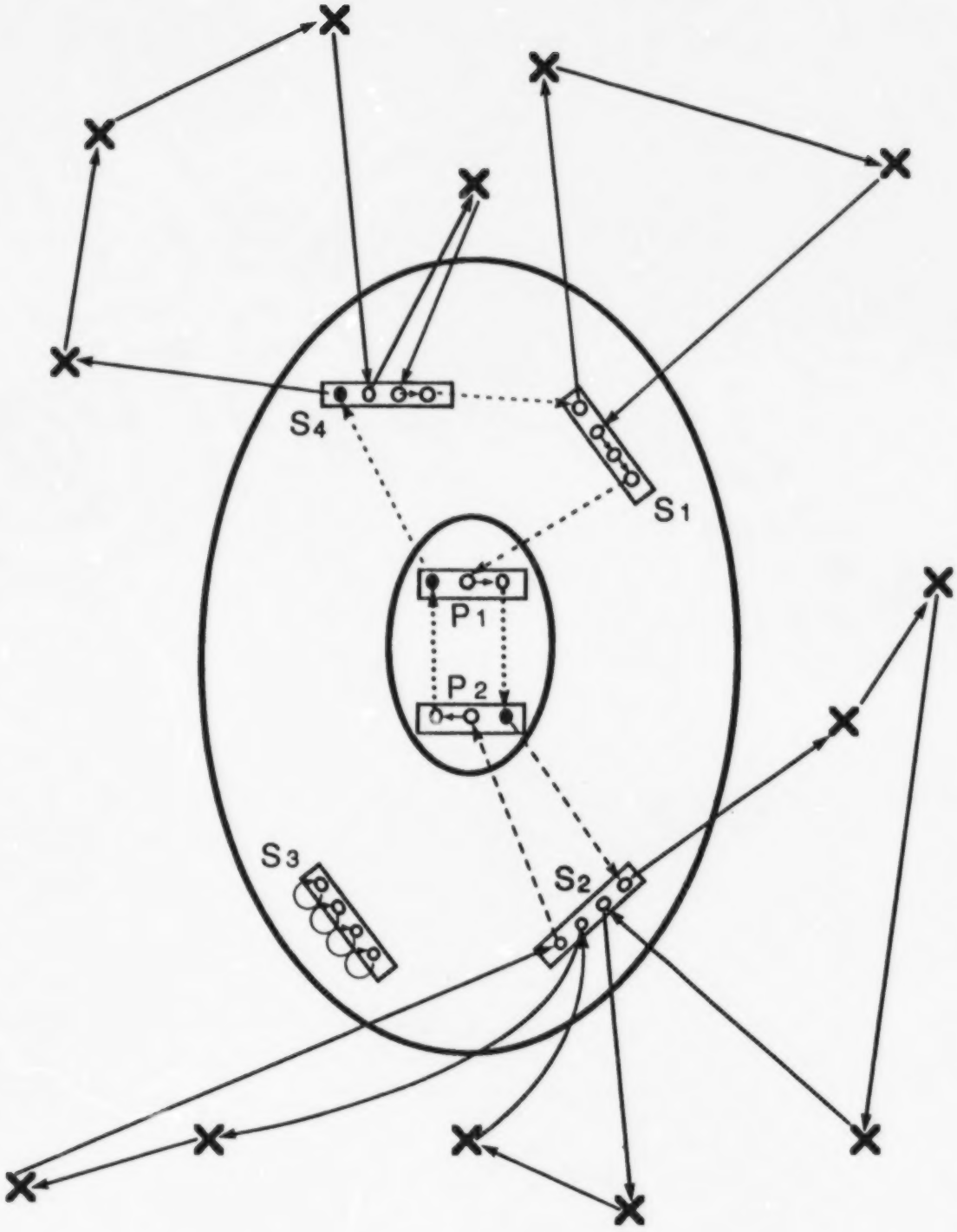

Figure 2

Feasible solution on G'

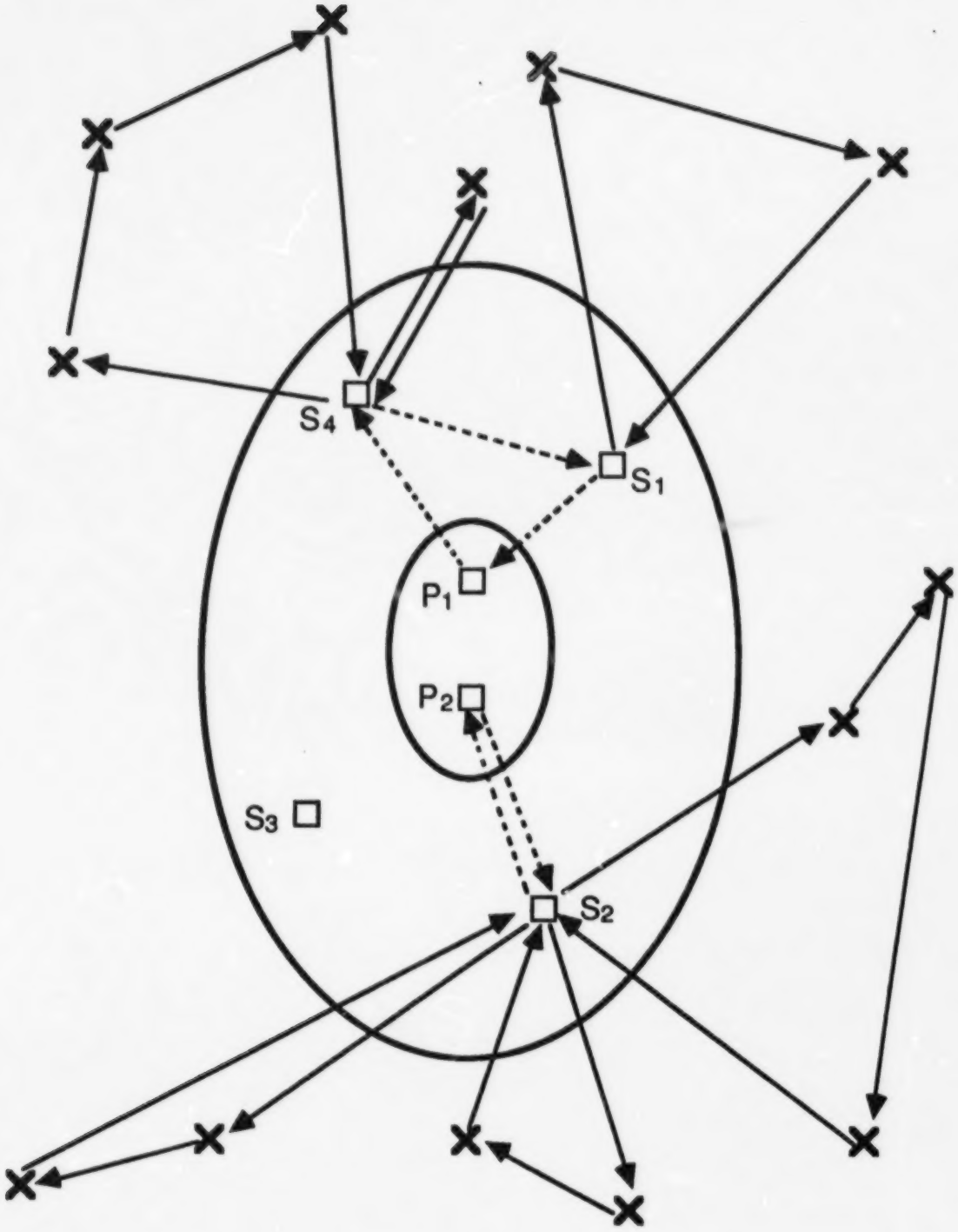

Figure 3

Feasible solution on G

In this formulation, constraints (54) and (55) specify that all nodes which appear in the solution must be entered and left once; if the variable x_{ii} associated with a node of N_2' is equal to 1, the node is unvisited and the sums of incoming and outgoing arcs are then equal to zero. Constraints (56) are subtour elimination constraints; they need not be defined for sets of S which are subsets of N_2'. Constraints (57), (58) and (60) are self-explanatory. Constraints (59) forbid solutions in which vehicles based at unopened facilities are used. These correspond to constraints (42) in (LRP5).

7.5 Algorithm

The problem obtained by relaxing constraints (56) to (59) is an assignment problem (AP) for which efficient algorithms exist (see the recent survey by Martello and Toth (1987)). A sensible approach is to solve (LRP6) by means of a branch and bound tree : at the first node, the AP relaxation of LRP6 is solved. Then, and at every subsequent node of the search tree, a check for violated relaxed constraints is made. If at least one such constraint can be identified, a number of subproblems are created according to the rule proposed by Carpaneto and Toth (1980), by appropriately fixing at 0 and at 1 the variables corresponding to an illegal subtour (constraints (56)) or route (constraints (57) to (59)) and by solving the constrained AP. The development of the search tree follows the usual branch and bound rules. Such an approach was used successfully by Laporte et al. (1986a, 1986b) for the capacitated VRP and for a class of asymmetrical VRPs. Tests on LRP's will be reported in a subsequent publication.

8. CONCLUSION

This survey has shown that LRPs are encountered in a variety of practical situations and that their study constitutes a fast growing research area. Most developments have occurred over the last few years and, if comparisons are made with either pure location problems or pure

VRPs, research on LRPs is relatively fragmented, often addresses problems which are too specific and contains several voids which have yet to be filled.

At the end of this study, we can identify a number of promising research areas :

i) **Development and systematic analysis of heuristics** for basic types of LRPs (following the work of Branco and Coelho(1986), for example).

ii) **Development of new types of exact algorithms.** So far, most exact algorithms for LRPs have been derived from two-index vehicle flow formulations, coupled with constraint relaxation. Methods based on Lagrangean relaxation techniques and on dynamic programming (using state-space relaxation, see Christofides et al. (1981b)) which have shown great success for the VRP have not yet been used for LRPs. They hold, in our opinion, a great potential.

iii) Location and routing constitute, in several practical contexts, decisions which must be made at different levels. The optimal siting of factories, warehouses and depots usually constitutes a long-term decision which requires heavy investment, whereas routing is operational in nature and may have to be revised frequently. In this spirit, the study of **hierarchical LRPs** along the lines of Marchetti Spaccamela et al. (1982), for example, would constitute a welcome contribution to the field.

iv) Similarly, the study of **dynamic location problems** in which vehicles would make round-trips rather than return trips, should be of major interest. Details on this family of problems can be found in Geoffrion and Powers (1980), Erlenkotter (1981) and Dejax (1986).

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